Lecture 29. Length and orthogonality of vectors

Def Consider a vector

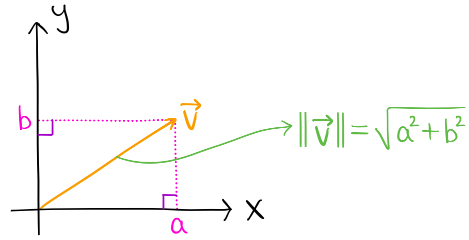
$$\vec{v} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

(1) The length (or norm) of $\vec{v}$ is

$$\boxed{\|\vec{v}\| := \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}}$$

(2) If $\vec{v}$ has length 1, it is called a unit vector.

Note The length of a vector in $\mathbb{R}^2$ comes from the Pythagorean theorem.



Prop Given a vector $\vec{v} \in \mathbb{R}^n$, a unit vector in the direction of $\vec{v}$ is

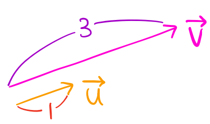
$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|}.$$

pf $\vec{u}$ is a multiple of $\vec{v} \Longrightarrow \vec{u}$ is in the direction of $\vec{v}$

$\vec{v}$ has length $\|\vec{v}\| \Longrightarrow \vec{u} = \dfrac{\vec{v}}{\|\vec{v}\|}$ has length $\dfrac{\|\vec{v}\|}{\|\vec{v}\|} = 1$

e.g. $\vec{v} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \Longrightarrow \|\vec{v}\| = \sqrt{2^2 + (-1)^2 + 2^2} = 3$

$\Longrightarrow \vec{u} = \frac{1}{3}\vec{v}$ is a unit vector in the direction of $\vec{v}$

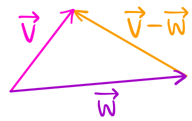


Def Consider two vectors

$$\vec{v} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \text{ and } \vec{w} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

(1) The distance between $\vec{v}$ and $\vec{w}$ is

$$\|\vec{v} - \vec{w}\| = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + \cdots + (a_n - b_n)^2}$$



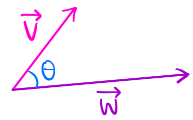
(2) The dot product (or inner product) of $\vec{v}$ and $\vec{w}$ is

$$\boxed{\vec{v} \cdot \vec{w} := \vec{v}^T \vec{w} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n}$$

(3) $\vec{v}$ and $\vec{w}$ are orthogonal if we have $\vec{v} \cdot \vec{w} = 0$

Note Alternatively, we may write

$$\vec{v} \cdot \vec{w} = \|\vec{v}\| \, \|\vec{w}\| \cos\theta$$



where θ is the angle between $\vec{v}$ and $\vec{w}$

$\vec{v}$ and $\vec{w}$ are orthogonal $\iff \vec{v} \cdot \vec{w} = 0 \iff \cos\theta = 0 \iff \theta = \frac{\pi}{2}$

Prop Given vectors $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$ the following identities hold.

(1) $\|\vec{v}\|^2 = \vec{v} \cdot \vec{v}$

(2) $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$

(3) $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$

Note We can use these identities for many algebraic computations

e.g. $(\vec{v} + \vec{w}) \cdot (\vec{v} - \vec{w}) = \vec{v} \cdot \vec{v} + \cancel{\vec{v} \cdot \vec{w}} - \cancel{\vec{w} \cdot \vec{v}} - \vec{w} \cdot \vec{w} = \|\vec{v}\|^2 - \|\vec{w}\|^2$

Ex Consider the vectors

$$\vec{v} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} \quad \text{and} \quad \vec{w} = \begin{bmatrix} 5 \\ -2 \\ 3 \end{bmatrix}$$

(1) Find the distance between $\vec{v}$ and $\vec{w}$

Sol $\|\vec{v} - \vec{w}\| = \sqrt{(3-5)^2 + (-1-(-2))^2 + (1-3)^2} = \sqrt{(-2)^2 + 1^2 + (-2)^2} = \sqrt{9} = \boxed{3}$

(2) Determine whether the vector

$$\vec{u} = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$$

is orthogonal to both $\vec{v}$ and $\vec{w}$.

Sol $\vec{u} \cdot \vec{v} = 1 \cdot 3 + 4 \cdot (-1) + 1 \cdot 1 = 0$

$\vec{u} \cdot \vec{w} = 1 \cdot 5 + 4 \cdot (-2) + 1 \cdot 3 = 0$

$\Rightarrow$ $\vec{u}$ is $\boxed{\text{orthogonal to both } \vec{v} \text{ and } \vec{w}}$

(3) Determine whether $\vec{u}$ is orthogonal to $4\vec{v} - 3\vec{w}$.

Sol $\vec{u} \cdot (4\vec{v} - 3\vec{w}) = 4\underbrace{\vec{u} \cdot \vec{v}}_{=0} - 3\underbrace{\vec{u} \cdot \vec{w}}_{=0} = 0$

$\Rightarrow$ $\vec{u}$ is $\boxed{\text{orthogonal to } 4\vec{v} - 3\vec{w}}$

Note In fact, we can apply the same argument to show that $\vec{u}$ is orthogonal to every linear combination of $\vec{v}$ and $\vec{w}$.

Ex Find a unit vector $\vec{u}$ which is orthogonal to the vectors

$$\vec{v} = \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix} \text{ and } \vec{w} = \begin{bmatrix} 4 \\ 1 \\ 8 \end{bmatrix}.$$

Sol We first find a vector $\vec{x}$ which is orthogonal to both $\vec{v}$ and $\vec{w}$.

We want $\vec{v} \cdot \vec{x} = 0$ and $\vec{v} \cdot \vec{x} = 0$

$\Longrightarrow 3x_1 + 2x_2 - 4x_3 = 0$ and $4x_1 + x_2 + 8x_3 = 0$

Hence we solve the equation $A\vec{x} = \vec{0}$ for

$$A = \begin{bmatrix} 3 & 2 & -4 \\ 4 & 1 & 8 \end{bmatrix} \text{ with } \mathrm{RREF}(A) = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -8 \end{bmatrix}.$$

* A has $\vec{v}$ and $\vec{w}$ as rows

$$\begin{cases} x_1 + 4x_3 = 0 \\ x_2 - 8x_3 = 0 \end{cases} \Longrightarrow \begin{cases} x_1 = -4x_3 \\ x_2 = 8x_3 \end{cases} \underset{x_3 = t}{\Longrightarrow} \vec{x} = t\begin{bmatrix} -4 \\ 8 \\ 1 \end{bmatrix}$$

Take $t = 1$: $\vec{x} = \begin{bmatrix} -4 \\ 8 \\ 1 \end{bmatrix} \Longrightarrow \|\vec{x}\| = \sqrt{(-4)^2 + 8^2 + 1^2} = 9$

$$\Longrightarrow \vec{u} = \frac{\vec{x}}{\|\vec{x}\|} = \boxed{\frac{1}{9}\begin{bmatrix} -4 \\ 8 \\ 1 \end{bmatrix}}$$

Note (1) We can take any nonzero value for t as it does not affect the direction of $\vec{x}$.

(2) Alternatively, we may use the cross product to find

$$\vec{u} = \frac{\vec{v} \times \vec{w}}{\|\vec{v} \times \vec{w}\|}$$

However, the cross product is defined only for $\mathbb{R}^3$.